Conformal positive mass theorems

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Abstract

We show the following two extensions of the standard positive mass theorem (one for either sign) : Let $(\mathcal{N}, g)$ and $(\mathcal{N}, g')$ be asymptotically flat Riemannian 3-manifolds with compact interior and finite mass, such that g and g' are $C^{2,\alpha}$ and related via the conformal rescaling $g' = \phi^4 g$ with a $C^{2,\alpha}-$ function $\phi > 0$. Assume further that the corresponding Ricci scalars satisfy $R \pm \phi^4 R' \geq 0$. Then the corresponding masses satisfy $m \pm m' \geq 0$. Moreover, in the case of the minus signs, equality holds iff g and g' are isometric, whereas for the plus signs equality holds iff both $(\mathcal{N}, g)$ and $(\mathcal{N}, g')$ are flat Euclidean spaces.

While the proof of the case with the minus signs is rather obvious, the case with the plus signs requires a subtle extension of Witten's proof of the standard positive mass theorem. The idea for this extension is due to Masood-ul-Alam who, in the course of an application, proved the rigidity part $m + m' = 0$ of this theorem, for a special conformal factor. We observe that Masood-ul-Alam's method extends to the general situation.

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The positive mass theorem of Schoen and Yau [20] and Witten [23] is a mathematical result with a direct physical interpretation: It showed that the concept of mass in relativity as defined in [1] is useful. Moreover, it has also proved to be an important tool in obtaining mathematical results of a more general nature within and beyond Relativity. In such applications the positive mass theorem has normally been used in combination with a suitable conformal rescaling of the metric, and it is in one way or the other important to keep control over the the mass in this process. This applies to the Yamabe problem [13], to Herzlich's proof of a Penrose-type inequality [10] and in particular to the uniqueness result for non-degenerate static black holes by Bunting and Masood-ul-Alam [3].

In these contexts the following two results (one for either sign) might be of interest. A special case has already been proven and applied before, as will be outlined below.

Theorem. Let $(\mathcal{N}, g)$ and $(\mathcal{N}, g')$ be asymptotically flat Riemannian 3-manifolds with compact interior and finite mass, such that g and g' are $C^{2,\alpha}$ and related via the conformal rescaling $g' = \phi^4 g$ with a $C^{2,\alpha}-$ function $\phi > 0$. Assume further that the corresponding Ricci scalars satisfy $R \pm \phi^4 R' \geq 0$.

Then the corresponding masses satisfy $m \pm m' \geq 0$. Moreover, in the case of the minus sign, equality holds iff g and g' are isometric, whereas for the plus sign equality holds iff both $(\mathcal{N}, g)$ and $(\mathcal{N}, g')$ are flat Euclidean spaces.

Due to their formal similarity we could not resist presenting these two results in a unified manner. However, their proofs as well as their interpretations and applications are quite different as far as presently known. We first discuss these interpretations and applications and postpone the technical part.

The bound on the mass given by the "-" part of the theorem has the following direct interpretation. Note that in Newtonian theory it is clear that the mass of a system exceeds the mass of another one if the density of matter of the first system exceeds the density of matter of the second system everywhere. In relativity one cannot expect such a subadditivity property to hold in general because the gravitational field also carries energy. Nevertheless, as matter density is represented on a time-symmetric slice by the Ricci scalar, the "-" part of the above theorem is a result of this kind. It may be interpreted by saying that, under conformal rescalings of time symmetric data, the change of their "matter component" always dominates the change in their "radiative component".

As to the "+" case of the theorem, it will clearly be meaningful only if non-positive masses are allowed, at least a priori. Then the result may be interpreted as above, and it might be relevant for quantum gravity. On the other hand, the rigidity part of this case has proved particularly interesting as a technical tool in uniqueness proofs for black holes, which we recall here.

Generalizing the classical result due to Israel [12], Bunting and Masood-ul-Alam proved that the Schwarzschild black hole solution is the unique static, asymptotically flat and appropriately regular vacuum spacetime with a non-degenerate Killing horizon [3]. In essence their method consists of performing, on the induced metric on the $t = const.$ slice, a suitable conformal transformation which removes the mass, followed by applying the rigidity case of the standard positive mass theorem. This result generalizes easily (namely by applying formally the same conformal rescaling as in the vacuum case) to show the absence of regular single scalar fields [25] or the absence of rather special $\sigma-$model fields [11] in spacetimes with non-degenerate horizons. With some effort a suitable conformal rescaling could also be obtained in the Einstein-Maxwell case, which yields uniqueness of the non-extreme Reissner-Nordström solution [15, 19, 22]. Furthermore, extensions of the vacuum and the electrostatic case with include extreme horizons have also been obtained [5, 6]. However, already in the coupled Einstein-Maxwell-dilaton case apparently natural conformal rescalings do not produce (manifestly) non-negative Ricci scalars as required for applying the standard version of the positive mass theorem.

This motivated Masood-ul-Alam [16] to prove the rigidity case of the "+"- part of the theorem above, for a particular conformal factor. He could then apply this result to show, in the Einstein-Maxwell dilaton case without magnetic fields and with non-degenerate horizons, the uniqueness of a 2-parameter family of solutions found by Gibbons [9].

The "+"-part of the theorem as formulated in this paper is useful for obtaining uniqueness proofs in more general situations, in particular for black holes in the presence of more general matter fields. Such results will be described elsewhere.

We finally remark here that it would also be desirable to obtain a "spacetime"-version of our result. This means that we expect to obtain corresponding bounds on the ADM 4-momentum [1] by imposing suitable requirements on the gravitational Cauchy data (g, p) and on suitably conformally rescaled data (g', p'). General results on the conformal behaviour of such data [24], and the "spacetime" formulation of the positive mass theorem as given in Witten's original paper [23] suggest that this might be possible.

We consider the following class of manifolds.

Definition. A Riemannian 3-manifold $(\mathcal{N}, g)$ is said to be asymptotically flat with compact interior and to satisfy the mass decay conditions (AFCIMD) if $\mathcal{N}$ is the union of a compact set $\mathcal{K}$ and an end $\mathcal{E}$ of topology R^3 minus a ball, and if (with respect to some asymptotic structure which we suppress in our notation)

$$g - \delta \in C^{2,\alpha}(\mathcal{N}) \cap C^{2,\alpha}_{-\tau}(\mathcal{E}) \qquad \text{for } 0 < \alpha < 1, \ \frac{1}{2} < \tau < 1, \text{ and} \tag{1}$$

$$R \in L^1(\mathcal{N}). \tag{2}$$

Here δ is the Kronecker symbol and $C^{2,\alpha}$ and $C^{2,\alpha}_{-\tau}$ denote Hölder spaces and weighted Hölder spaces respectively. For the latter we adopt the weight index convention of Bartnik [2] (also chosen by Lee and Parker [13]) which gives directly the growth at infinity, i.e. $f \in C^{2,\alpha}_{-\tau}$ for some function f implies $f = O(r^{-\tau})$ (and corresponding falloff conditions on the derivatives).

Some remarks on this AFCIMD definition are in order. The requirement $\tau < 1$ is not a restriction here but just introduces a notation suitable to formulate the lemma below. Note in particular that (1) does allow $g - \delta$ to fall off like $O(r^{-1})$. Thus (apart from this subtlety) our definition agrees with that required for a Witten-type proof in (the appendix of) [13]. The name "mass decay condition" is adopted from Bartnik (c.f. Def. 2.1 and Sect. 4 of [2]) who requires, however, weaker decay conditions formulated in terms of Sobolev spaces. The reason for formulating the present work in terms of Hölder spaces is again the lemma on the uniqueness of the conformal structure given below, which then becomes a rather obvious consequence of a known result.

Both within the Hölder as well as within the Sobolev setting, the AFCIMD conditions are the weakest ones which guarantee that the ADM mass

$$m = \frac{1}{16\pi} \int_{S_\infty} (\partial_j g_{ij} - \partial_i g_{jj}) dS^i \tag{3}$$

is well defined and finite [2, 7]. (Here and below, dS^i denotes the outward normal surface element to S_∞, the sphere at infinity, and repeated indices are summed over).

For what follows it is useful to recall the behaviour of the Ricci scalar under conformal rescalings $g' = \phi^4 g$, viz.

$$\triangle\phi = \frac{1}{8}(R - \phi^4 R')\phi. \tag{4}$$

We have the following lemma on uniqueness of the conformal structure.

Lemma. Let $(\mathcal{N}, g)$ be a Riemannian manifold which satisfies the AFCIMD conditions as formulated in the definition above. Then the same applies to $(\mathcal{N}, g')$ with $g' = \phi^4 g$ iff

$$\begin{aligned} \phi - 1 &\in C^{2,\alpha}(\mathcal{N}) \cap C^{2,\alpha}_{-\tau}(\mathcal{E}) \qquad \text{for } 0 < \alpha < 1,\ \frac{1}{2} < \tau < 1, \text{ and} && (5)\\ \triangle\phi &\in L^1. && (6) \end{aligned}$$

Proof. The result that $(\mathcal{N}, g')$ is AFCIMD follows trivially from (5) and (6), using (4).

On the other hand, requiring that $(\mathcal{N}, g')$ is AFCIMD, (5) is a consequence of Theorem 2.4 of [8]. Then (6) is obvious from (2) and (4). □

A well known (or from (3) and (5) easily proven) fact is that the masses of two AFCIMD metrics g and $g' = \phi^4 g$ are related by

$$m - m' = \frac{1}{2\pi}\int_{S_\infty} \nabla_i\phi \; dS^i. \tag{7}$$

Proof of the theorem, part "-". Applying Gauss' law to (4) and using (5) we can write (7) as

$$m - m' = \frac{1}{16\pi}\int_{\mathcal{N}} (R - \phi^4 R')\phi \; dV \tag{8}$$

where dV is the volume element on $(\mathcal{N}, g)$. The inequality $m - m' \geq 0$ is then obvious from the assumption that $R - \phi^4 R' \geq 0$. Requiring now that $m = m'$, eqn. (8) and the assumption on the Ricci scalars imply $R = \phi^4 R'$ on $\mathcal{N}$. Thus, from (4) we have $\triangle\phi = 0$ on $\mathcal{N}$. But since $\phi \in C^{2,\alpha}$ and goes to 1 at infinity, this is only possible if $\phi = 1$ on $\mathcal{N}$, which had to be shown. $\square$

The "+" part of the theorem will now be shown via Witten's techniques, as an extension of the proof of Masood-ul-Alam [15].

We consider the bundle of Dirac spinors Γ as recalled, e.g. in [2, 13, 18], denote by $C^{2,\alpha}(\Gamma)$ and by $C^{2,\alpha}_{-\tau}(\Gamma)$ the Hölder spaces and weighted Hölder spaces of sections of Γ, respectively, and by $\mathcal{D}$ the Dirac operator. We adopt the notation $\sigma_{ij} = \frac{1}{2}[e_i, e_j] = e_i e_j + \delta_{ij}$ for the Clifford algebra with basis e_i.

To facilitate understanding of the following manipulations we recall the so-called Lichnerowicz identity for $\Psi \in C^{2,\alpha}(\Gamma)$ (which is in fact due to Schrödinger, formula (74) of [21]),

$$\mathcal{D}^2\Psi = \nabla * \nabla\Psi + \frac{1}{4}R\Psi. \tag{9}$$

where $\nabla * \nabla = -g^{ij}\nabla_i\nabla_j = -\nabla^i\nabla_i$ is the covariant Laplacian on spinors. This relation implies, for solutions of the Dirac equation $\mathcal{D}\Psi = 0$,

$$\triangle|\Psi|^2 = \frac{1}{2}R|\Psi|^2 + 2|\nabla\Psi|^2. \tag{10}$$

Again for solutions of the Dirac equation we then find, using (4) and (10) in the final step,

$$\begin{aligned}
&\nabla_i[\nabla^i|\Psi|^2 - 2\phi^{-1}(\nabla_i\phi)|\Psi|^2] = \\
&\quad = \triangle|\Psi|^2 + 2\phi^{-2}(\nabla_i\phi)(\nabla^i\phi)|\Psi|^2 - 2\phi^{-1}(\triangle\phi)|\Psi|^2 - 2\phi^{-1}(\nabla^i\phi)\nabla_i|\Psi|^2 = \\
&\quad = \frac{1}{4}(R + \phi^4 R')|\Psi|^2 + 2|\nabla_i\Psi - \phi^{-1}(\nabla_i\phi)\Psi|^2.
\end{aligned} \tag{11}$$

Proof of the theorem, part "+". In analogy with [2] and [15], we show first that the Dirac operator

$$\mathcal{D} : C^{2,\alpha}_{-\tau}(\Gamma) \to C^{1,\alpha}_{-\tau-1}(\Gamma) \qquad \text{for } 0 < \alpha < 1,\ 0 < \tau < 2 \tag{12}$$

is an isomorphism. Passing to Sobolev spaces $W^{1,q}_{-\epsilon}$ (as defined e.g. in [2]) via the embedding $C^{k,\alpha}_{-\tau}(\Gamma) \subset W^{k,q}_{-\epsilon}(\Gamma)$ for any $k \geq 0$, $q > 1$, $\epsilon < \tau$, standard results ([4, 14]) imply that, for $q > 1$, $0 < \epsilon < 2$,

$$\mathcal{D} : W^{2,q}_{-\epsilon}(\Gamma) \to W^{1,q}_{-\epsilon-1}(\Gamma) \tag{13}$$

is Fredholm with adjoint

$$\mathcal{D} = \mathcal{D}^* : W^{2,q'}_{\epsilon+1-n}(\Gamma) \to W^{1,q'}_{\epsilon-n}(\Gamma) \tag{14}$$

for $q' = (1 - q^{-1})^{-1}$. If $\Psi \in \ker \mathcal{D}$, then $|\Psi| \to 0$ at infinity. The strong maximum principle applied to relation (11) then shows that $|\Psi|^2 = 0$. Hence (13) and its adjoint have trivial kernels and are isomorphisms. The regularity claimed in (12) then follows from the ellipticity of the Dirac operator [17].

Let now Ψ_0 be a spinor which is constant at infinity. Then there is a spinor Ψ such that

$$\begin{aligned} \mathcal{D}\Psi &= 0, & (15) \\ \Psi - \Psi_0 &\in C^{2,\alpha}_{-\tau}(\Gamma). & (16) \end{aligned}$$

We can thus find a solution Ψ of the Dirac equation which tends to a prescribed constant spinor. We now integrate (11) over a ball with boundary S_ρ (a sphere of coordinate radius ρ), use the requirement $R + \phi^4 R' \geq 0$ of the theorem and apply Gauss' law to obtain

$$\begin{aligned} 0 \;\leq\; & \int_{S_\rho} [\nabla_i |\Psi|^2 - 2|\Psi|^2 \nabla_i \ln\phi] dS^i = \\ & = 2\int_{S_\rho} (\langle \Psi, \nabla_i \Psi\rangle - |\Psi|^2 \nabla_i \ln\phi) dS^i = \\ & = 2\int_{S_\rho} (\langle \Psi, \sigma_{ij} \cdot \nabla_j \Psi\rangle - |\Psi|^2 \nabla_i \ln\phi) dS^i. \end{aligned} \tag{17}$$

Finally, passing to the limit $\rho \to \infty$, the first surface integral in (17) is known to give the mass [13], whereas the second one is evaluated by virtue of (5) and (7). This yields

$$0 \leq 8\pi |\Psi_0|^2 m - 4\pi |\Psi_0|^2 (m - m') = 4\pi |\Psi_0|^2 (m + m'). \tag{18}$$

To show the rigidity case we note that using $m + m' = 0$ in the integral of (11) yields $\nabla(\phi^{-1}\Psi) = 0$, $R = 0$ and $R' = 0$. The existence of a covariantly constant spinor implies by a standard argument (see e.g. [2]) that $(\mathcal{N}, g)$ is flat, so in particular $m = 0$. Therefore we also have $m' = 0$, and applying the standard positive mass theorem on $(\mathcal{N}, g')$ finishes the proof. □

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